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Factorization of finite strains in three dimensions—a computer method

A. M. BELL

Geological Survey of Ireland, 14 Hume Street, Dublin, Eire

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Abstract—Strain analysis using deformed objects is already well established. The more sophisticated techniques can resolve tectonic strains and indicate the nature of any pre-tectonic fabric. Methods that analyse in two dimensions can produce problems of incompatability when results from three faces of one sample are combined.

A method for factorizing finite, non-coaxial strains is presented which overcomes these problems by analysing in three dimensions. Published data from deformed lapilli tuff from the English Lake District have been used to test the method.

Results are as valid as those obtained originally from the data, and the technique enables strain analysis to be extended to areas which give imprecise results using existing methods.

PREVIOUS METHODS

STRAIN analysis using deformed objects and fossils is now well established as a tool available to the structural geologist. Early work by Ramsay (1967) has been developed by several authors (Dunnett 1969, Elliott 1970, Dunnett & Siddans 1971, Matthews *et al.* 1974). Sophisticated two-dimensional analytical methods have been described which can resolve tectonic strains and identify various pre-strain fabrics (Dunnett & Siddans 1971). Matthews *et al.* (1974) provide a statistical check on the precision of results. Oertel (1970) has presented a three-dimensional analysis of the strain recorded by a lapillar tuff from the English Lake District. All these methods attempt to resolve strain ellipses or ellipsoids from final deformed shapes.

In both two-dimensional and three-dimensional methods, the magnitude and direction of the final shape is measured or calculated. The amount of data known about the components now controls the level of success any factorization can achieve. If both the magnitude and orientation of the latest tectonic effect are known, the pre-strain ellipsoid may be determined in full and can be further factorized if necessary. This situation is unlikely, since determining the magnitude of this ellipsoid is usually the object of the exercise. If neither the magnitude nor the direction of the tectonic effect is known, no unique solution is possible: there are an infinite number of valid factorizations with no obvious method of selecting relevant values.

When the direction but not the magnitude of the strain ellipsoid is known a suite of results again exists, but solutions can be found by selecting strain magnitudes in a systematic manner. Data in this form are most commonly used for strain analyses. The best strain magnitude can be isolated by reference to the quality of the factorization. This may be determined statistically if pre-strain ellipses possessed a distinctive orientation. Alternatively, a particular pre-strain fabric may be predicted from other geological evidence. The sequential selection of strain values and inspection of results forms the basis of many graphical and computerised strain analyses.

There are advantages in performing an analysis in two dimensions. Most sections through material to be used for this purpose will show more than one deformed object, so iterative routines can be used to examine the shapes of each particle as increments of strain are removed. Statistical parameters can then help to identify acceptable pre-strain fabrics and stop the iterative process. This forms the basis of the computerised Rf/φ technique described by Dunnett & Siddans (1971). This approach is not generally applicable to three-dimensional methods since three sections through individual objects cannot be obtained unless the objects can be removed whole for analysis.

The usefulness of two-dimensional analyses can unfortunately be marred by practical problems. To produce results for one face of a sample using these methods is straightforward and accurate, but it is possible to factorize incompatible pre-strain data from the three separate faces of one sample. The present author used the Rf/φ method to analyse nine samples of cleaved accretionary-lapilli tuff from the Borrowdale Group of the English Lake District (Bell 1975, p. 39). Analysis of one specimen from Wrengill Quarry showed that a face cut parallel to the cleavage plane, XY, and one cut in the YZ plane both showed symmetric Rf/φ plots. The XZ face produced an asymmetric plot which, on unstraining, indicated a semi-planar fabric. It is unlikely that two sections with symmetric diagrams could be found perpendicular to one another unless the sample possessed a random pre-strain fabric. This result is, however, not compatible with that from the third face. If the sample suffered no significant pre-strain deformation, the XZ face, which records the greatest tectonic strain, should have shown the most symmetric rather than the least symmetric Rf/φ diagram. Internal inconsistencies of this sort occurred with seven of the nine specimens mentioned above. One specimen produced a random, a planar and an imbricate initial fabric from three mutually perpendicular faces.

It is likely that in many samples from the southeast Lake District the closeness of strike of the bedding to that of the cleavage plane, together with the steep plunge of the principal extension direction on the cleavage face (Soper & Numan 1974), have produced low precision in the Rf/φ method. Dunnett & Siddans (1971, p. 316) have already noted that this method is unreliable where a bedding trace is sub-parallel to a principal strain direction.

There is advantage to be gained from combining final ellipse data into an ellipsoid prior to analysis if internally consistent factorizations can be made, but pre-strain fabrics must be inferred from geological evidence. For example, diagenetic compaction will have occurred in indurated sedimentary rocks, and a planar pre-strain fabric aligned in the bedding plane (Oertel & Curtis 1972) may have been formed. The recognition of such a fabric may be used to stop an iterative routine; this approach has been used in the method described here. In addition, some indication of the precision of results may be gained if error values are carried through the analysis; a technique used by Oertel (1970).

Oertel performed his three-dimensional analysis on a trial-and-error basis, using a desk calculator. This paper presents a computerised systematic method for the factorization of ellipsoids in three dimensions.

ASSUMPTIONS

The method requires that a final deformation ellipsoid should be measurable, both in magnitude and orientation. A tectonic frame must be defined. Limited information about the pre-strain ellipsoid must be available. In this respect, the assumptions required by this model are no more than the three-dimensional equivalents of those that accompany the Rf/φ method. As the model is presently set up, the unstraining process is considered complete when an oblate pre-strain ellipsoid whose short axis lies normal to the pre-strain bedding plane is obtained. Thus the model is most suitable for factorizing a single strain ellipsoid from rocks which have undergone compaction during diagenesis, followed by one penetrative deformation. The model assumes that the final ellipsoid is representative of the strain the sample has suffered. All ellipsoids are expressed assuming constancy of volume.

THE METHOD

A computer programme has been written in FORTRAN IV to perform the factorization of the measured final ellipsoid into ellipsoids oriented in a tectonic strain frame and a pre-strain frame. Listings are available as an open-file report of the Geological Survey of Ireland. In the test case presented here, Oertel's (1970) published data are used, data being derived from an accretionary-lapilli tuff cut by a slaty cleavage. The cleavage plane, X_s Y_s, and the extension lineation within it, X_s, have been used to define the tectonic frame. The bedding plane

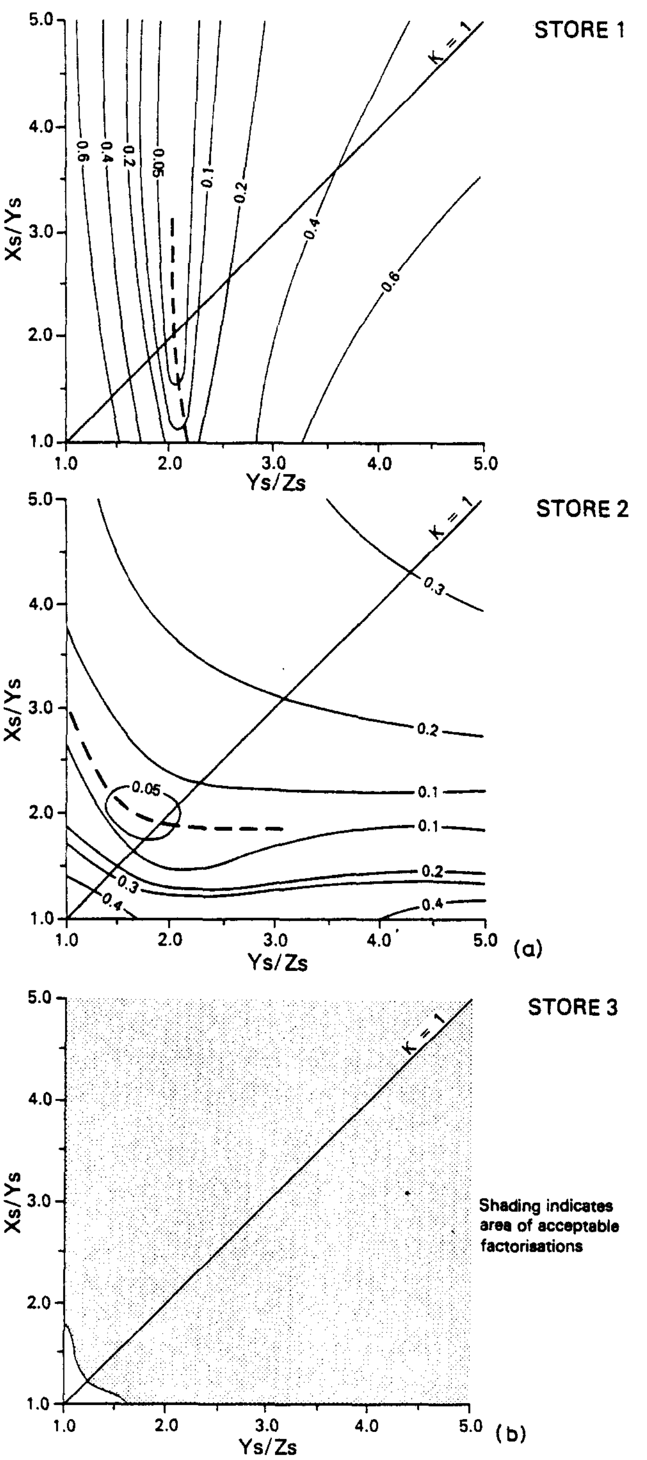

Fig. 1. STORE 1, STORE 2 and STORE 3 after factorization. (a) Contoured Flinn plots showing oblateness of the pre-strain ellipsoid (STORE 1) and its orientation relative to the pre-strain bedding plane (STORE 2). The intersection of minimum curves (broken lines) locates the tectonic ellipsoid. (b) Flinn plot showing area of valid factorizations (STORE 3).

defines an orientation for the pre-strain ellipsoid X_i, Y_i, Z_i. Throughout the calculation the cleavage frame is used as a reference frame.

Average final ellipses for three mutually perpendicular faces are calculated from two-dimensional data using programme STRANE presented by Dunnett & Siddans (1971). The method described by Ramsay (1967, section 4.7) has been used to combine these ellipses into a final ellipsoid, although refinements to this technique now exist, such as that developed by W. H. Owens (presented to the *Annual Meeting of the Tectonic Studies Group*, Liverpool, 1978). This ellipsoid is represented in matrix form and rotated from its principal reference frame into the tectonic frame using matrix versions of equations 3–18 of Ramsay (1967).

The strain analysis is then performed in two parts. Firstly, the final ellipsoid is unstrained using Ramsay's (1967, p. 92) equations, again in matrix form. A pre-strain ellipsoid is obtained by the systematic superimposition of various reciprocal strain ellipsoids on the final shape. For this initial analysis, 81 strain ellipsoids are chosen by a systematic scan of the Flinn plot (X_s/Y_s = 1.0, 1.5, 2.0, . . ., 5.0; Y_s/Z_s = 1.0, 1.5, 2.0, ..., 5.0) and these strains removed in turn from the final ellipsoid. The bedding plane is also restored to a pre-strain orientation using equations 4.21 of Ramsay (1967). Results of this scan are recorded in STORE 1, STORE 2 and STORE 3 (Figs. 1a & b).

STORE 1 records a measurement of the difference in length between the two axes of the pre-strain ellipsoid that lie in the pre-strain bedding plane, contoured from the 81 data points described above. STORE 2 records the sum of the absolute values of the off-diagonal elements of the representation matrix of the pre-strain ellipsoid, and represents the alignment of the pre-strain ellipsoid with respect to the pre-strain bedding plane. STORE 3 indicates the orientation of the shortest axis of the pre-strain ellipsoid. If that short axis lies approximately normal to the pre-strain bedding plane, STORE 3 records a value of zero. If, however, the short axis lies closer to the bedding plane, warning values are given. Since results occupy the same location in STORE as the strain ellipsoid used for unstraining occupies on its Flinn plot, a number other than zero in a particular site in STORE 3 indicates that the tectonic strain represented by that site is incapable of producing a satisfactory factorization.

The three STORE tables give direct information about the nature of the factorizations for a large range of strains. For example, if a strain ellipsoid whose Flinn parameters are $X_s/Y_s = 1.5$; $Y_s/Z_s = 2.0$ is removed from the final shape (Fig. 1), the parameter representing the difference in length of the two axes which lie in the restored bedding plane (STORE 1) has a value of 0.049. STORE 3 indicates that, in this site, these two axes are indeed X_i and Y_i. If a more prolate strain ellipsoid is removed (reading upwards keeps Y_s/Z_s = 2.0, but increases X_s/Y_s through 1.5, 2.0, 2.5, . . . to 5.0), the difference between the long and intermediate axes of the pre-strain ellipsoid decreases, but the orientation of that ellipsoid (STORE 2), although at first improving, quickly becomes less well aligned in the pre-strain bedding frame. When the higher strains are removed, a more oblate pre-strain ellipsoid is factorized, although it is less well oriented in the bedding than it was after the removal of a strain of $X_s/Y_s = 2.0$. Therefore, the lower the combined values of STORE 1 and STORE 2, the closer that strain is to providing the best factorization. Figure 1 represents contoured values of STORE 1 and STORE 2 for this test case. It can be seen that a minimum occurs in both tables near the value $X_s/Y_s = Y_s/Z_s = 2.0$.

A more accurate determination of this minimum is made in the second part of the strain analysis. The rows of the Flinn plot are scanned at intervals of 0.1 X_s/Y_s for STORE 1 and the columns scanned at intervals of 0.1 Y_s/Z_s for STORE 2 to locate accurately the 'valleys' in Fig. 1. The point of intersection of the two 'valleys' locates the tectonic ellipsoid which, when removed from the final ellipsoid, gives a pre-strain factor which is almost oblate and whose axes lie close to the bedding frame. This is found by selecting a polynomial which best fits the minimum points defining the 'valley' of each STORE. Cubics have been fitted to avoid the unwanted points of inflexion generated by polynomials of greater orders. These curves are indicated by dashed lines in Fig. 1. The point at which the two cubics intersect locates the required tectonic ellipsoid. In the programme, this is achieved by the standard technique of equating the two polynomials and finding the roots of the resulting cubic.

Table 1. Summary of input data and results from the present method. Orientations relative to computing frame. Input data after Oertel (1970)

	Flinn parameters			Axial orientations						Bedding data			
				X		*Y*		*Z*		Pole to *S'*		Pole to *S*	
	X/Y	*Y/Z*	*K*	*PL*	*AZ*	*PL*	*AZ*	*PL*	*AZ*	*PL*	*AZ*	*PL*	*AZ*
Final Ellipsoid	1.35	1.71	0.50	02	136	06	225	84	025	18	201		
Resolved ellipsoids: ROOT 1 Tectonic	1.87	2.06	0.82	00	180	00	270	90	000				
ROOT 1 pre-strain	1.00	2.06	0.00	62	110	27	279	04	011			05	012
ROOT 2 Tectonic	1.51	2.11	0.46	00	180	00	270	90	000				
ROOT 2 pre-strain	1.02	1.71	0.03	72	125	17	285	06	017			06	014
ROOT 3 Tectonic	Out	of	range										

Three roots exist, of which only one is meaningful to the strain analysis. It frequently happens, though, that one or both of the other roots lie within the range of the Flinn plot.

For each root within the range, the strain analysis is repeated and the pre-strain matrix and restored bedding plane re-calculated. Each pre-strain matrix is then diagonalized to obtain its true magnitude and orientation, and the matrix, together with its principal axial lengths and orientations, printed out. The strain value which gives the best results may then be selected by inspection.

A summary of the output from the programme for this test case is given in Table 1. A successful factorization has been achieved (Root 1). The resolved pre-strain ellipsoid is oblate to within two decimal places and the short axis lies within half a degree of the pole to the pre-strain bedding plane. Figure 2 shows the magnitude and true orientation of the final, strain and pre-strain ellipsoids. The pre-strain ellipsoid is oriented in the pre-strain bedding frame for all practical purposes (Fig. 2b).

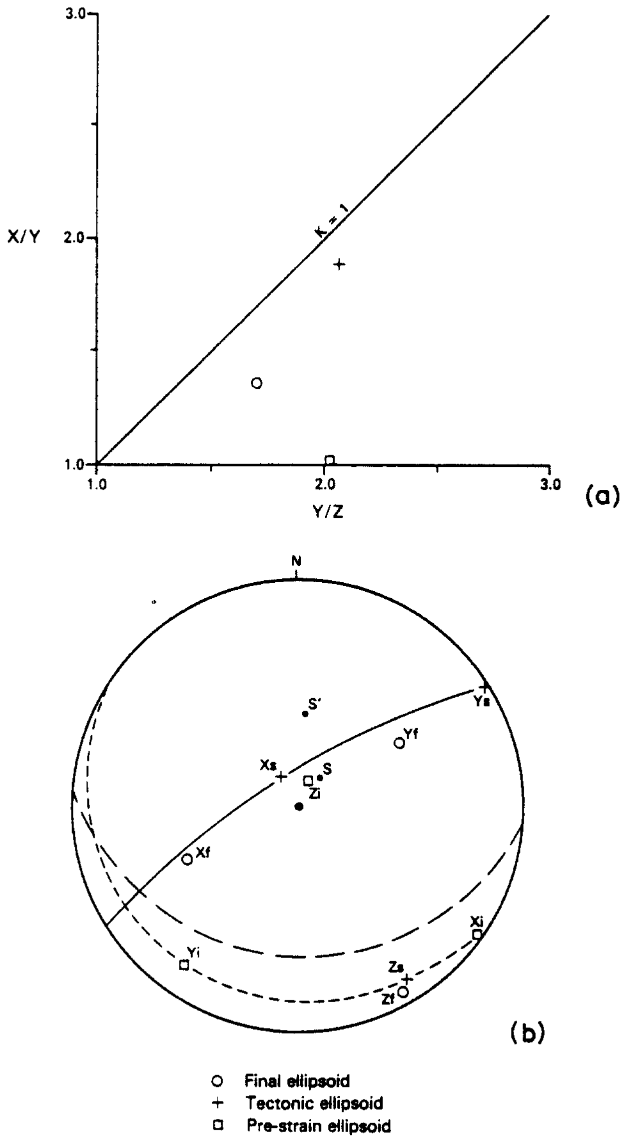

Fig. 2. Magnitude and actual orientation of component ellipsoids. (a) Flinn plot. (b) Equal-area, lower hemisphere projection. Ellipsoids—$X > Y > Z$. Cleavage plane—solid line. Deformed bedding—long broken line (pole S'). Pre-cleavage bedding plane—short broken line (pole S).

DISCUSSION

This result is as acceptable a factorization as that made originally by Oertel (1970, p. 1182), but the tectonic ellipsoid determined by the present method is significantly different from Oertel's (X_s/Y_s = 1.865, Y_s/Z_s = 2.056, compared with $X_s/Y_s = Y_s/Z_s$ = 2.0 by Oertel). The more prolate value of Oertel's cleavage ellipsoid (K= 1.000 against K= 0.818 from the present method) requires the cofactorization of an oblate spheroid of greater axial ratios than that produced here.

The existence of two distinct, but equally acceptable, factorizations requires closer examination. Figure 1 shows that cleavage ellipsoids with one particular Y_s/Z_s value (Y_s/Z_s= 2.0) produce oblate pre-strain ellipsoids for most stages of this process. The 'best' cleavage ellipsoid is controlled by the orientation of that oblate spheroid. A zone of cleavage strains exists in which the oblate spheroid lies almost in the bedding frame. This zone extends from X_s/Y_s = 1.87 to X_s/Y_s = 2.00. There is, therefore, a set of results which fall within the acceptable error limits. All give similar values for both strain and pre-strain ellipsoids, but results within this zone, which factorize more prolate strain ellipsoids, also factorize more deformed pre-strain ellipsoids. In general, methods of analysis that scan in intervals, whether computerised or not, are unlikely to locate an exact answer and only allow a close approximation to the exact answer to be found. The errors here are, however, of a similar order to those incorporated during the measurement of the deformed objects initially, so may be included within acceptable observational limits. In this context it is generally unwise to lay too much emphasis on the actual numerical value produced by strain analysis.

The above method allows only two component ellipsoids to be factorized. In many situations the rock has suffered more than two deformations, in others the measured particles may have had a significant pre-compaction shape. Additional strains, or particle shape variation, may themselves be represented by ellipsoids. These effects will have been factorized into components and included into the values for the two primary factors. It should not, therefore, be assumed that the resolved ellipsoids represent actual strain or pre-strain ellipsoids in every case. Care should be taken in interpreting results from methods of this kind, and an input of extra geological information is almost always necessary.

Finally, no factorization method that selects magnitude values for strain ellipsoids with reference to the pre-strain shape is capable of producing more than two components. It is not acceptable to extend the analysis by subjecting either component to refactorization.

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